A note on the divergence-free Jacobian Conjecture in $\mathbb{R}^2$ *

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Abstract

We give a shorter proof to a recent result by Neuberger [11], in the real case. Our result is essentially an application of the global asymptotic stability Jacobian Conjecture. We also extend some of the results presented in [11].

1 Introduction

The classical Jacobian Conjecture was formulated in [8] as a problem about the global invertibility of polynomial maps $\Phi : \mathbb{C}^n \mapsto\to \mathbb{C}^n$. Keller asked whether a polynomial map with constant non-zero Jacobian determinant is globally invertible, and its inverse is itself a polynomial map. The problem was widely studied in subsequent decades, producing several partial results and even some faulty proofs. In [1] one finds a historical overview of research about the Jacobian Conjecture and a rich survey of results published up to

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1982. The paper [2] contains a more recent list of results and some equivalent formulations of the problem in arbitrary dimension. Among general results concerning such a problem, it is known that it is equivalent to prove or disprove the statement in any field of zero characteristic, that it is sufficient to prove Φ's injectivity in order to get its surjectivity [1], and that Φ's global invertibility implies that Φ^{-1} is a polynomial map. The most studied special case is the bidimensional one, $\Phi(x, y) = (P(x, y), Q(x, y))$, where the statement was proved under the hypothesis that either P's or Q's degree is 4, or prime, or both degrees are ≤ 100 (see [1] for a more comprehensive list of results). A recent result, which is the object of this paper, proves the global invertibility of jacobian maps of the form $\Phi(x, y) = (x + p(x, y), y + q(x, y))$, $p(x, y)$ and $q(x, y)$ without terms of degree 1, under the additional assumptions that $\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} = 0$ and $\frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x} = 0$. In higher dimensions, a striking result states that, in order to prove the n-dimensional Jacobian conjecture, it is sufficient to prove it for maps of the form $\Phi = L + C$, L linear, C cubic, [1], or even for maps of the form $\Phi(X) = X + (AX)^3$, where A is a nilpotent matrix [4].

A different question, arising in differential equations from the study of a critical point's global stability, is also known as a Jacobian Conjecture. It is concerned with the global asymptotic stability (g. a. s.) of a critical point of a vector field whose jacobian eigenvalues have negative real part at every point of the space [10]. In [12] it was showed that under such hypotheses, it is equivalent to prove the global asymptotic stability of a critical point or the global injectivity of the vector field. Such a result gave a new direction to the research about the g. a. s. Jacobian Conjecture. Thanks also to such a new approach, such a question was positively settled in dimension 2 in [5], [6], [7]. In higher dimensions it is known to be false [3], unless some additional hypotheses hold.

In this paper we give a shorter proof to Neuberger's result [11] in the real case, showing that it is actually a consequence of the bidimensional g. a. s. Jacobian Conjecture. Actually, we prove something more, since we do not make assumptions on the terms in $\frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}$. Actually, for jacobian maps it is sufficient to require that $\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \geq 0$. Then we look for algebraic-like conditions which imply such a property, involving the degree and the order of the real polynomials P and Q, or the degrees of the monomials contained in P and Q. We also extend some of the corollaries proved in [11], weakening

some symmetry conditions.

2 Results

Througout this paper we only consider polynomials with real coefficients. Given a polynomial P, we write $d(P)$ for its degree, $o(P)$ for its order. We say that a polynomial is *even* if it is the sum of even-degree monomials, *odd* if it is the sum of odd-degree monomials. Similalrly, we say that a polynomial is *x-even* if it contains only terms with even powers of x, *x-odd* if it contains only terms with odd powers of x.

We say that a non-negative integer is a *gap* of P if it is the difference of the degrees of two distinct monomials in P. We denote by $G(P)$ the gap-set of P. As an example, the polynomial $P(x,y) = x^3+y^3+x^2y^2+y^7$ has gap-set $G(P) = \{0,1,3,4\}$. If P has exactly one monomial, then we say that it has empty gap-set.

We say that the couple of polynomials (P,Q) satisfies the *gap condition* if for every monomial M in P, one has $d(M)-1 \notin G(Q)$. The gap condition is not symmetric, as shown by the couple $(P,Q) = (x+y^2, x^6+y^2)$. In such a case one has $G(P) = \{1\}$, $G(Q) = \{4\}$, so that (P,Q) satisfies the gap condition, but (Q,P) does not.

We say that (P,Q) satisfies the *symmetric gap condition* if both (P,Q) and (Q,P) satisfy the gap condition.

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(x,y) \equiv (P(x,y), Q(x,y))$ be a real polynomial map. Let J_Φ be its jacobian matrix. We say that Φ is a *jacobian map* if its jacobian determinant $\det J_\Phi$ is a non-zero constant. We first consider a straightforward consequence of the g. a. s. Jacobian Conjecture.

Lemma 1 *Let $\Phi^* : \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi^*(u,v) = (u + p^*(u,v), v + q^*(u,v))$ be a jacobian polynomial map, with $o(p^*) > 1$, $o(q^*) > 1$. If $\frac{\partial p^*}{\partial u} + \frac{\partial q^*}{\partial v} \geq 0$, then Φ^* is injective.*

Proof. Since $\det J_{\Phi^*}$ is constant, its value can be evaluated at the origin, hence $\det J_{\Phi^*} = 1$. Let us consider the planar differential system associated

to the map $-\Phi^*$,

$$\dot{u} = -u - p^*(u,v), \qquad \dot{v} = -v - q^*(u,v).$$

Its jacobian matrix is $-J_{\Phi^*}$. Since $\frac{\partial p^*}{\partial u} + \frac{\partial q^*}{\partial v} \geq 0$, at every point of the plane $-J_{\Phi^*}$ has trace $\leq -2 < 0$, and determinant $\det(-J_{\Phi^*}) = \det J_{\Phi^*} = 1 > 0$, hence its eigenvalues have negative real part at every point of $\mathbb{R}^2$. Since the g. a. s. Jacbian Conjecture holds, $-\Phi^*$ is injective, hence Φ^* is injective, too. ♣

Neuberger's result for real maps is contained in lemma 1, since in [11] only maps with $\frac{\partial p^*}{\partial u} + \frac{\partial q^*}{\partial v} = 0$ are considered.

In relation to the classical Jacobian Conjecture, algebraic-like hypotheses are usually considered. In fact, even if checking whether $\frac{\partial p^*}{\partial u} + \frac{\partial q^*}{\partial v} \geq 0$ in some cases can be done, statements related to the map's degree or order are desirable.

Condition *ii*) of next theorem has been added only because we do not assume the linear part of Φ to be the identity, but the argument is the same as in [11]. The other conditions, as well as those ones in theorem 2, are new.

Theorem 1 *Let* $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ *be a jacobian map of the type* $\Phi(x,y) = (ax + by + p(x,y), cx + dy + q(x,y))$, $a,b,c,d \in \mathbb{R}$, $o(p) > 1$, $o(q) > 1$. *If one of the following holds,*

i) $\max\{d(p), d(q)\} < o(p) + o(q) - 1$,

ii) both $p(x,y)$ *and* $q(x,y)$ *are even polynomials,*

iii) p *is odd,* q *is even and* (p,q) *satisfies the gap condition,*

iv) (p,q) *satisfies the symmetric gap condition,*

then Φ *is globally invertible.*

Proof. Without loss of generality, we may assume $ad - bc > 0$. Let A be the linear map associated to the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let A^{-1} be its inverse. Let us set $\Phi^*(u,v) = \Phi(A^{-1}(u,v))$. The linear part of Φ^* is just the composition of A^{-1} and A, hence it is the identity. Then,

one has $\Phi^*(u,v) = (u + p^*(u,v), v + q^*(u,v))$, with $\det J_{\Phi^*} > 0$. A linear change of variables does not change a polynomial's order, its degree and the property of being even or odd, as above defined. Hence $o(p^*) > 1$, $o(q^*) > 1$, and conditions $i)$, ..., $iv)$ hold for p^* and q^* as well. Moreover, one has $\det J_{\Phi^*} > 0$. Without loss of generality we may assume $\det J_{\Phi^*} = 1$.

Computing the jacobian determinant of Φ^* gives

$$\det J_{\Phi^*} = 1 + \left(\frac{\partial p^*}{\partial u} + \frac{\partial q^*}{\partial v} \right) + \left(\frac{\partial p^*}{\partial u} \frac{\partial q^*}{\partial v} - \frac{\partial p^*}{\partial v} \frac{\partial q^*}{\partial u} \right).$$

Let us set

$$T^* = \frac{\partial p^*}{\partial u} + \frac{\partial q^*}{\partial v}, \qquad D^* = \frac{\partial p^*}{\partial u} \frac{\partial q^*}{\partial v} - \frac{\partial p^*}{\partial v} \frac{\partial q^*}{\partial u}.$$

Since $o(T^* + D^*) > 0$, one has $T^* + D^* \equiv 0$.

In order to prove $i)$, consider that the highest degree monomial in T^* has degree $\leq \max\{d(p), d(q)\} - 1$, while the lowest degree monomial in D^* has degree $\geq o(p) + o(q) - 2$. If $\max\{d(p), d(q)\} < o(p) + o(q) - 1$, then T^* and D^* have no monomials of the same degree, hence, from $T^* + D^* \equiv 0$, one gets both $T^* \equiv 0$ and $D^* \equiv 0$. Applying the lemma 1 one proves that $-\Phi^*$ is injective, hence Φ^* and Φ are injective, too.

Now, in order to prove $ii)$, consider that if both $p(x,y)$ and $q(x,y)$ are even polynomials, then $p^*(x,y)$ and $q^*(x,y)$ are even, T^* is odd and D^* is even. From $T^* + D^* \equiv 0$, one gets again $T^* \equiv 0$ and $D^* \equiv 0$, since monomials of T^* do not cancel with monomials of D^*. Then one can proceed as in point $i)$ for the injectivity of Φ.

Under the hypotheses of $iii)$, $\frac{\partial p^*}{\partial u}$ is even, $\frac{\partial q^*}{\partial u}$ is odd, D^* is odd. Assume by absurd that there exists a positive integer h such that both $\frac{\partial q^*}{\partial u}$ and D^* have a monomial of degree h. Then q has a monomial M such that $d(M) - 1 = h$. Also, there exist monomials K in p and L in q such that $d(K) + d(L) - 2 = h$. Hence $d(M) - d(L) = d(K) - 1$, contradicting the gap condition. This proves that T^* and D^* have no monomials of the same degree, so that $T^* \equiv 0$ and $D^* \equiv 0$. Then the above argument applies.

Finally, if $iv)$ holds, assume by absurd that there exists a positive integer h such that both T^* and D^* have a monomial of degree h. Then, either p or q has a monomial M such that $d(M) - 1 = h$. If M is in q, we may repeat

the argument of point iii). If M is in p, we may repeat the argument of point iii), exchanging the roles of p and q. ♣

In order to show that we are considering non-empty hypotheses, we give some examples of jacobian mappings satisfying the above conditions. For condition i) we may choose Meisters' maps,

$$\Phi(x,y) = (ax+by+\mu(\alpha a+\beta b)(\alpha y-\beta x)^2, cx+dy+\mu(\alpha c+\beta d)(\alpha y-\beta x)^2),$$

with $\mu \neq 0$, $(\alpha,\beta) \neq (0,0)$, $ad-bc \neq 0$. For condition ii) we may consider

$$\Phi(x,y) = (x+y+x^5+x^6, y+x^5+x^6).$$

An example of map satisfying both conditions iii) and iv) is

$$\Phi(x,y) = (x+y^3, y).$$

In [1] it was proved that in order to prove the Jacobian Conjecture it is sufficient to prove it for cubic-linear jacobian maps. It may be interesting to show that in $\mathbb{R}^2$ every jacobian map of the type linear + homogeneous is invertible.

Corollary 1 *Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a jacobian map of the type $\Phi(x,y) = (ax+by+p_n(x,y), cx+dy+q_n(x,y))$, $a,b,c,d \in \mathbb{R}$, with p_n and q_n homogeneous polynomials of the same degree $n > 1$. Then Φ is globally invertible.*

Proof. One has $o(p) = o(q) = d(p) = d(q) = n$, hence condition i) is satisfied: $\max\{d(p), d(q)\} = n < 2n-1 = o(p)+o(q)-1$. ♣

An example of planar linear + cubic jacobian map is the following map,

$$P(x,y) = 2x-y+x^3+x^2y+\frac{xy^2}{3}+\frac{y^3}{27}, \quad Q := 3x-3y+\frac{12x^3}{5}+\frac{12x^2y}{5}+\frac{4xy^2}{5}+\frac{4y^3}{45}.$$

A result slightly different form theorem 1 can be proved by assuming different symmetries on p and q. In next statement we assume Φ to be of the form $(x+p(x,y), y+q(x,y))$, as in [11], since a linear change of variables in general does not preserve the requested symmetry property.

Theorem 2 *Let* $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ *be a jacobian map of the type* $\Phi(x,y) = (x + p(x,y), y + q(x,y))$, $o(p) > 1$, $o(q) > 1$. *If one of the following holds,*

i) p *is* x*-even,* q *is* x*-odd,*

ii) p *is* y*-odd,* q *is* y*-even,*

then Φ *is globally invertible.*

Proof. Working as in theorem 1, one has

$$\det J_\Phi = 1 + \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) + \left(\frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}\right).$$

Let us set

$$T = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}, \qquad D = \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}.$$

As in theorem 1, since $o(T + D) > 0$, one has $T + D \equiv 0$.

If $i)$ holds, then T is x-odd, D is x-even. Hence terms of T and D cannot cancel with each other, and both T and D vanish identically. Then one can proceed as in the proof of theorem 1.

If $ii)$ holds, then T is y-odd, D is y-even. Then one can proceed as above. ♣

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